Thermal Modulation of Rayleigh-Benard Convection

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Thermal convection in a fluid layer confined between two horizontal rigid boundaries has been studied with the help of the Floquet theory. The temperature distribution consists of a steady part and an oscillatory time-dependent part. Disturbances are assumed to be infinitesimal. Numerical results for the critical Rayleigh numbers and wave numbers are obtained. It is found that the disturbances are either synchronous with the primary temperature field or have half its frequency. Some comparisons have also been made.

Key words: Modulation; Stability; Rayleigh Number; Galerkin Method; Thermal Convection.

1. Introduction

Thermal convective instability often arises when a fluid is heated from below. This instability can occur in the atmosphere and the ocean. Thermal convection in a horizontal fluid layer heated from below has been studied extensively during the last fifty years. Chandrasekhar [1] has given a comprehensive review of this stability problem. Drazin and Reid [2], Koschmieder [3] and Getling [4] have also discussed in detail about Rayleigh-Benard convection.

The present paper deals with the thermal convection in a fluid layer confined between two horizontal rigid planes. The fluid layer is heated from below and above periodically with time. In 1969 Venezian [5] investigated this problem as the thermal analogue of Donnelly's experiment [6]. Venezian's theory does not find any such finite frequency, as obtained by Donnelly, but finds that for the case of modulation only at the lower surface, the modulation would be stabilizing with maximum stabilization occurring as the frequency goes to zero.

Considering low-frequency modulation, Rosenblat and Herbert [7] have investigated the linear stability problem and found an asymptotic solution. They considered free-free surfaces and employed periodicity and amplitude criteria to calculate the critical Rayleigh number. Rosenblat and Tanaka [8] have investigated the problem by using the Galerkin procedure. However, in their analysis they considered rigid-rigid boundaries. Yih and Li [9] have investigated the formation of convective cells in a horizontal fluid layer, confined between two rigid boundaries. They found that the disturbances (or convection cells) oscillate either synchronously or with half frequency.

Gresho and Sani [10] have treated the linear stability problem with rigid boundaries. They found that gravitational modulation could significantly affect the stability limits of the system. Finucane and Kelly [11] have carried out an analytical-experimental investigation to confirm the results of Rosenblat and Herbert. Besides investigating the linear stability, Roppo et al. [12] have also carried out the weakly non-linear analysis of the problem. Weimin and Charles [13] have obtained a numerical solution of the linear Rayleigh-Benard convection and compared the results with the analytic solution.

Aniss et al. [14] have carried out an investigation on linear problem of the convection parametric instability in the case of a Newtonian fluid confined in a Hele-Shaw cell and subjected to a vertical periodic motion. In their asymptotic analysis they have investigated the influence of the gravitational modulation on the instability threshold. Bhatia and Bhadauria [15, 16], Bhadauria and Bhatia [17] have studied the linear stability problem for more general temperature profiles. Recently Bhadauria et. al. [18] have studied the thermal convection in a Hele-Shaw cell with parametric excitation under saw-tooth and step-function oscillations.

The objective of the present study is to find the critical conditions under which thermal convection

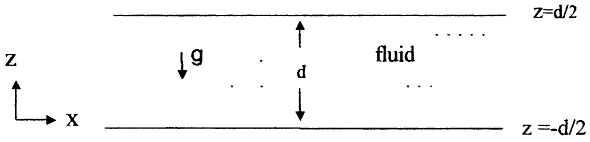

Fig. 1. Benard Configuration.

starts. A sinusoidal profile is considered to modulate the wall temperatures, and the effect of modulation on the thermal convective instability is investigated. Galerkin's method is used to solve the problem. The results have their relevance with convective flows in the terrestrial atmosphere.

2. Formulation

We consider a layer of a viscous, incompressible fluid, confined between two parallel horizontal walls, one at $z = -d/2$ and the other at $z = d/2$. The walls are infinitely extended and rigid. The configuration is shown in the Figure 1.

The governing equations in the Boussinesq approximation are

$$\frac{\partial \boldsymbol{V}}{\partial t}+\boldsymbol{V}\cdot\nabla\boldsymbol{V} = -\frac{1}{\rho_0}\nabla p+\alpha g(T-T_0)\hat{k}+\nu\nabla^2\boldsymbol{V}, \tag{2.1}$$

$$\nabla\cdot\boldsymbol{V} = 0, \tag{2.2}$$

$$\frac{\partial T}{\partial t}\boldsymbol{V}\cdot\nabla T = \kappa\nabla^2 T, \tag{2.3}$$

where ρ_0 and T_0 are the constant reference density and temperature, respectively, g is the acceleration due to gravity, ν the kinematic viscosity, κ the thermal diffusivity, α the coefficient of volume expansion, and $\boldsymbol{V} = (u, v, w)$ the fluid velocity. The relation between ρ_0 and T_0 is given by

$$\rho = \rho_0[1 - \alpha(T - T_0)]. \tag{2.4}$$

Here we consider the modulation of the temperature of both the lower and upper boundary. The externally imposed wall temperatures are:

$$\overline{T} = \begin{cases} T_0 - T_2\cos(\omega' t) & \text{at} \quad x_3 = -\dfrac{d}{2}, \\ T_1 - T_2\cos(\omega' t) & \text{at} \quad x_3 = \dfrac{d}{2}, \end{cases} \tag{2.5}$$

where ω' is the modulating frequency and $2\pi/\omega'$ is the period of oscillation. The equations (2.1 - 2.4) along with the boundary conditions (2.5) admit an equilibrium solution in which

$$\boldsymbol{V} = (u, v, w) = 0, \ T = \overline{T}(z, t), \ p = \overline{p}(z, t). \tag{2.6}$$

An equation for the pressure $p = \overline{p}(z, t)$, which balances the buoyancy force, is not required explicitly, however the temperature $T = \overline{T}(z, t)$ can be given by the diffusion equation

$$\frac{\partial \overline{T}}{\partial t} = \kappa\frac{\partial^2 \overline{T}}{\partial z^2}. \tag{2.7}$$

The differential equation (2.7) can be solved with the help of the boundary conditions (2.5).

Here my objective is to examine the behaviour of infinitesimal disturbances to the basic solution (2.6). With this in view, substitute

$$\boldsymbol{V}' = (u', v', w'), \ T = \overline{T} + \theta', \ p = \overline{p} + p' \tag{2.8}$$

into (2.1 - 2.3) and linearize the equations with respect to the perturbation quantities $\boldsymbol{V}'$, θ'. Now, to simplify these equations we proceed as follows:

(i) The disturbance equations are reduced to the equations one for θ' and the other one for the velocity component w'.

(ii) These quantities are Fourier analyzed as

$$\begin{aligned} w'(x_1, x_2, x_3, t) &= w'(x_3, t)\exp[i(a_1' x_1 + a_2' x_2)], \\ \theta'(x_1, x_2, x_3, t) &= \theta'(x_3, t)\exp[i(a_1' x_1 + a_2' x_2)], \end{aligned} \tag{2.9}$$

where a_1' and a_2' are the wave numbers in the x_1 and x_2 direction, respectively.

(iii) All quantities are non-dimensionalized, we put

$$\begin{aligned} &(x, y, z) = \frac{1}{d}(x_1, x_2, x_3), \ \tau = \frac{t\kappa}{d^2}, \\ &V = \frac{dV'}{\kappa}, \ \omega = \omega'\frac{d^2}{\kappa}, \ a_1'^2 + a_2'^2 = d^2 a^2, \\ &\theta' = (T_0 - T_1)\theta, \ T = \left(\overline{T} - T_1\right)/\left(T_0 - T_1\right), \end{aligned} \tag{2.10}$$

where a is the non-dimensionalized horizontal wave number.

Then the boundary conditions (2.5) reduce to

$$T = \begin{cases} 1 - b\cos\omega\tau & \text{at} \quad z = -\dfrac{1}{2}, \\ b\cos\omega\tau & \text{at} \quad z = \dfrac{1}{2}, \end{cases} \tag{2.11}$$

where $b = T_2/(T_0 - T_1)$.

Then the solution of (2.7) can be written as

$$T = \frac{1}{2} - z + b\mathrm{Re}\left\{\frac{\sinh(\lambda z)}{\sinh(\lambda/2)} e^{i\omega\tau}\right\}, \qquad (2.12)$$

$$\text{where } \lambda^2 = i\omega. \qquad (2.13)$$

Also, the perturbation equations obtained after the processes (i), (ii) and (iii) are found to be

$$\frac{1}{P}\left(\frac{\partial^2}{\partial z^2} - a^2\right)\frac{\partial w}{\partial \tau} = -Ra^2\theta + \left(\frac{\partial^2}{\partial z^2} - a^2\right)^2 w, \qquad (2.14)$$

$$\frac{\partial \theta}{\partial \tau} = \left(\frac{\partial^2}{\partial z^2} - a^2\right)\theta - \left(\frac{\partial T}{\partial z}\right) w, \qquad (2.15)$$

where $R = \alpha g(T_0 - T_1)d^3/\nu\kappa$ is the Rayleigh number, $P = \nu/\kappa$ is the Prandtl number and $\partial T/\partial z$ is the temperature gradient which can be obtained directly from the equation (2.12). The boundary conditions on w and θ are

$$w = \frac{\partial w}{\partial z} = 0 \quad \text{at} \quad z = \pm\frac{1}{2}, \qquad (2.16)$$

$$\theta = 0 \quad \text{at} \quad z = \pm\frac{1}{2}. \qquad (2.17)$$

3. Method

The presence of the time-dependent coefficients in (2.15) prohibits a separation of variables solution to the system (2.14, 2.15). Therefore we use Galerkin's technique, which transforms the two partial differential equations (2.14) and (2.15) into a system of ordinary differential equations. The latter are then solved numerically. Here we have a set of linearly independent functions, which are called the basis functions. The dependent variables of the problem are represented by the finite sum of these basis functions. We put

$$w(z,\tau) = \sum_{m=1}^{N} A_m(\tau)\varphi_m(z), \qquad (3.1)$$

$$\theta(z,\tau) = \sum_{m=1}^{N} B_m(\tau)\psi_m(z), \qquad (3.2)$$

$$\text{where } \psi_m(z) = \sqrt{2}\sin m\pi\left(z + \frac{1}{2}\right) \qquad (3.3)$$

$$\text{and } \varphi_m(z) = \begin{cases} \dfrac{\cosh\mu_m z}{\cosh\mu_m/2} - \dfrac{\cos\mu_m z}{\cos\mu_m/2}, & \text{if } m \text{ is odd}, \\[2ex] \dfrac{\sinh\mu_m z}{\sinh\mu_m/2} - \dfrac{\sin\mu_m z}{\sin\mu_m/2}, & \text{if } m \text{ is even}. \end{cases} \qquad (3.4)$$

The functions $\psi_m(z)$ and $\varphi_m(z)$ above are defined in a manner so that they vanish at $z = \pm 1/2$. For their derivatives to vanish at these boundaries, it is required that μ_m are to be the roots of the characteristic equation

$$\tanh\frac{1}{2}\mu_m - (-1)^m \tan\frac{1}{2}\mu_m = 0. \qquad (3.5)$$

The roots of (3.5) are given in the book of Chandrasekhar ([1], p. 636).

It is clear that the functions $\psi_m(z)$ and $\varphi_m(z)$ each form an orthonormal set in the interval (–1/2, 1/2). Also they are even in (–1/2, 1/2) when m is odd, and vice versa.

Now we substitute (3.1) and (3.2) into (2.14) and (2.15). Then multiply (2.14) by $\varphi_m(z)$ and (2.15) by $\psi_n(z)$, $n = 1, 2, 3, ..., N$. The resulting equations are then integrated with respect to z in the interval (–1/2, 1/2). The outcome is a system of $L = 2N$, ordinary differential equations for the unknown coefficients $A_n(\tau)$ and $B_n(\tau)$.

$$\frac{1}{P}\sum_{m=1}^{N}\left[K_{mn} - a^2\delta_{mn}\right]\frac{\mathrm{d}A_m}{\mathrm{d}\tau} = -Ra^2\sum_{m=1}^{N} B_m(\tau)P_{mn} + \sum_{m=1}^{N}\left[\left(\mu_m^2 + a^4\right)\delta_{mn} - 2a^2K_{mn}\right]A_m(\tau), \qquad (3.6)$$

$$\frac{\mathrm{d}B_n}{\mathrm{d}\tau} = -(n^2\pi^2 + a^2)B_n(\tau) + \sum_{m=1}^{N}\left[P_{mn} - b\mathrm{Re}\left\{F_{mn}e^{i\omega\tau}\right\}\right]A_m(\tau) \quad (n = 1, 2, ..., N), \qquad (3.7)$$

where δ_{mn} is the Kronecker delta.

The other coefficients, which occur in (3.6) and (3.7) are

$$P_{mn} = \int_{-1/2}^{1/2}\psi_m(z)\varphi_n(z)\,\mathrm{d}z, \qquad (3.8)$$

$$K_{mn} = \int_{-1/2}^{1/2} \frac{\mathrm{d}^2\varphi_m(z)}{\mathrm{d}z^2}\varphi_n(z)\,\mathrm{d}z, \qquad (3.9)$$

and

$$F_{mn} = \frac{\lambda}{\sinh(\lambda/2)} \int_{-1/2}^{1/2} \cosh(\lambda z)\varphi_m(z)\psi_n(z)\,\mathrm{d}z. \qquad (3.10)$$

All these coefficients can be easily evaluated in their closed form, but here we have calculated them numerically using Simpson's (1/3)rd rule (Sastry [19], p. 125).

It is convenient for computational purpose to take $m = 1, 2, ..., N$, i. e. total 2N equations and then rearrange them. For this, first multiply the equations (3.6) by the inverse of the matrix $(K_{mn} - a^2\delta_{mn})$, and then introduce the notations

$$x_1 = A_1,\ x_2 = B_1,\ x_3 = A_2,\ x_4 = B_2, \ldots . \qquad (3.11)$$

Now combine (3.6) and (3.7) to the form

$$\frac{\mathrm{d}x_i}{\mathrm{d}\tau} = H_{i1}x_1 + H_{i2}x_2 + ... + H_{iL}x_L \qquad (3.12)$$

$$(i = 1, 2, 3, ..., 2N \text{ and } L = 2N),$$

where $H_{ij}(\tau)$is the matrix of the coefficients in (3.6) and (3.7).

4. Analysis

It is clear that the coefficients $H_{ij}(\tau)$ of (3.12) are either constant or periodic in τ with period $\tau_0 = 2\pi/\omega$, therefore the stability of the solution of (3.12) can be discussed by the Floquet theory (Cesari [20], p. 55). We know that the solution in the Floquet theory is of the form $e^{\mu\tau}P(\tau)$, where $P(\tau)$ is a periodic function of τ with period τ_0. The vanishing of the real part μ_r of μ gives the stability boundary. Here our aim is to determine the condition under which $\mu_r = 0$.

If the imaginary part μ_i of μ is zero, the disturbance is synchronous with the unsteady part of the mean temperature field. However if $\mu_i\tau_0$ is equal to π or $-\pi$, then the disturbance is subharmonic, having a frequency half that of the unsteady mean temperature field.

Let

$$x_n(\tau) = x_{in}(\tau) = \mathrm{col}\left[x_{1n}(\tau), x_{2n}(\tau), ..., x_{Ln}(\tau)\right]$$

$$(n = 1, 2, 3, ..., 2N) \qquad (4.1)$$

be the solutions of (3.12), which satisfy the initial conditions

$$x_{in}(0) = \delta_{in}. \qquad (4.2)$$

The solutions (4.1) with the conditions (4.2) form $2N$ linearly independent solutions of (3.12). Once these solutions are found, one can get the values of $x_{in}(2\pi)$, and then arrange them in the constant matrix

$$C = [x_{in}(2\pi)]. \qquad (4.3)$$

The eigenvalues λ_1, λ_2, λ_3, ... ,λ_L of the matrix C are also called the characteristic multipliers of the system (3.12), and the numbers μ_r, defined by the relations

$$\lambda_r = \exp(2\pi\mu_r),\ r = 1, 2, 3, ..., 2N, \qquad (4.4)$$

are the characteristic exponents.

The values of the characteristic exponents determine the stability of the system. We assume that the μ_r are ordered so that

$$\mathrm{Re}(\mu_1) \geq \mathrm{Re}(\mu_2) \geq ... \geq \mathrm{Re}(\mu_L). \qquad (4.5)$$

Then the system is stable if $\mathrm{Re}(\mu_1) < 0$, while $\mathrm{Re}(\mu_1) = 0$ corresponds to one periodic solution and represents a stability boundary. This periodic disturbance is the only disturbance which will manifest itself at marginal stability.

To obtain the matrix C we have integrated the system (3.12), using the Runge-Kutta-Gill Procedure (Sastry [19], p. 217). The eigenvalues λ_1, λ_2, λ_3, ... ,λ_L of the matrix C are found with the help of Rutishauser method (Jain, Iyengar, and Jain [21], p. 116).

5. Results and Discussion

As usual in the Galerkin procedure, we determine the value of N in the process of numerical solution. It is found that for the parameter ranges of our interest it is sufficient to take $N = 2$ (two Galerkin terms-one even and one odd). Therefore all the following results are related to $N = 2$. In this case the critical value of the Rayleigh number and the wave number in the absence of modulation ($b = 0$) are found as

$$R_\mathrm{c} = 1728.3859, a = 3.10. \qquad (5.1)$$

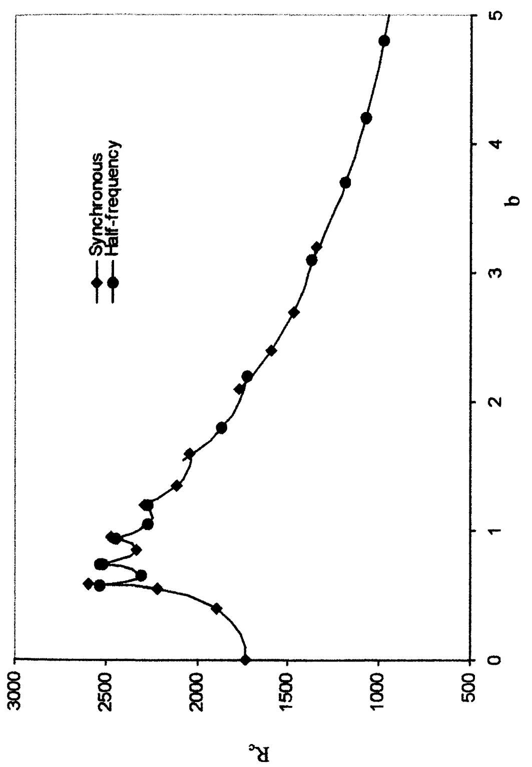

Fig. 2. Variation of R_c with respect to b. $P = 0.73$, $\omega = 5$.

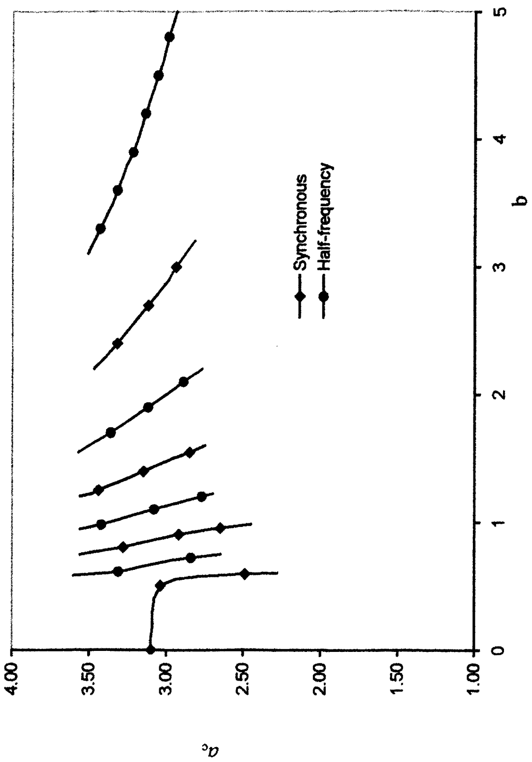

Fig. 3. Variation of a_c with respect to b. $P = 0.73$, $\omega = 5$.

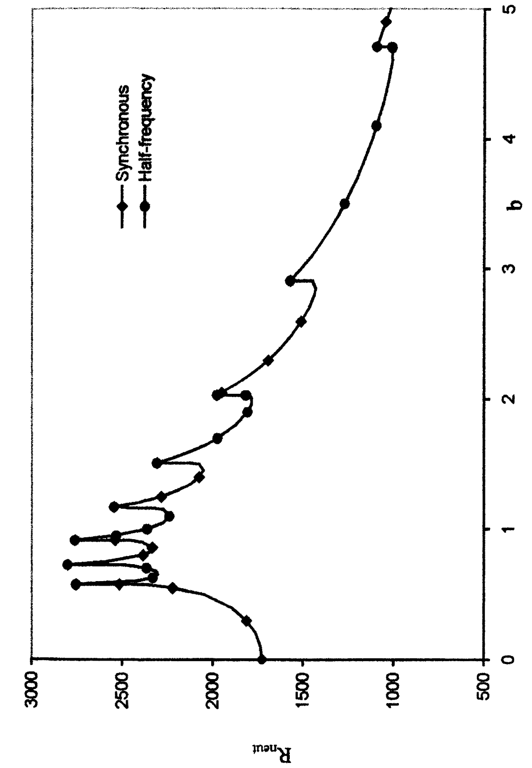

Fig. 4. Variation of R_{neut} with respect to b. $P = 0.73$, $\omega = 5$.

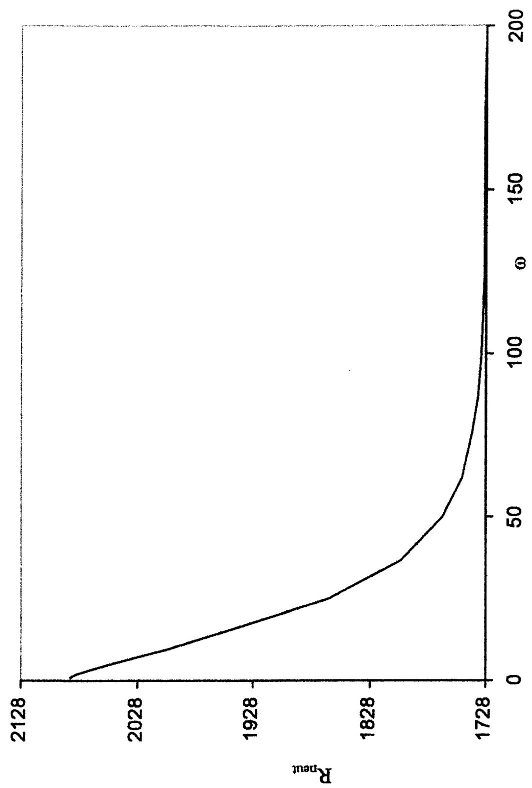

Fig. 5. Variation of R_{neut} with respect to ω. $P = 0.73$, $b = 0.5$.

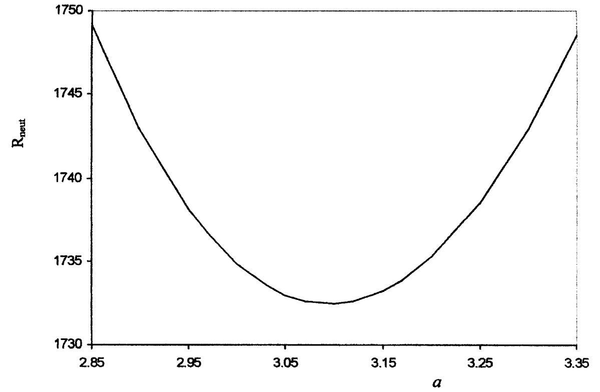

Fig. 6. Variation of R_{neut} with respect to a. $P = 0.73$, $b = 0.5$, $\omega = 100$.

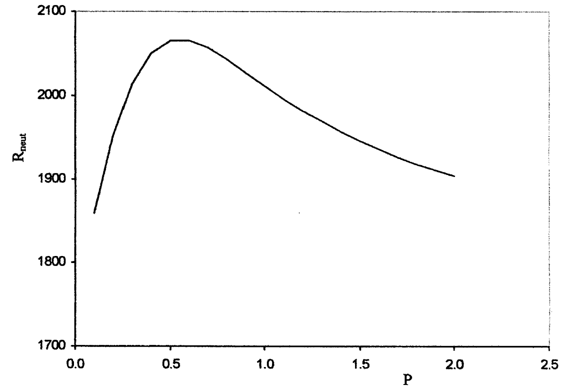

Fig. 7. Variation of R_{neut} with respect to P. $a = 3.1$, $b = 0.5$, $\omega = 5$.

When $b \neq 0$ and $T_0 \neq T_1$, we calculate the modified value of R_c with variation in other parameters. We also check the critical value of the wavenumber a. Here the results have been obtained by solving (3.12) for x_1, x_2, x_3 and x_4, The results are calculated at moderate values of b as we are interested only in the modulating effect of the oscillation; there seems to be no reason why this theory can not be applied for large values of the parameters.

Here it is possible to obtain a relationship between the critical Rayleigh number R_c and the corresponding critical wave number a_c in terms of the dimensionless frequency ω of the oscillating primary temperature field. Since the critical Rayleigh number is the minimum value of R as a function of the wave number a for a fixed value of ω, the critical curve consists of the different curves, corresponding to synchronous solutions (S-curve) and subharmonic solutions (H-curve).

In Fig. 2, we depict the variation of the critical Rayleigh number R_c with b. The graph consists S-curves and H-curves, which are alternative. From the figure one can see that each cusp in the R_c-b curve is really the intersection of an S-curve and an H-curve, both of which can be continued beyond their intersections. Thus in the area above the H-curve there are also synchronous disturbances, but disturbances with half-frequency can be expected to be more unstable. Similarly above an S-curve there are disturbances with half frequency, but synchronous disturbances are more unstable.

In the above figure we note that when b is small, the effect of modulation on the stability of the system is stabilizing with the convection occurring at a later point than in the unmodulated case ($b = 0$). However as b increases the effect of the unsteady part of the primary temperature field becomes destabilizing. The results agree very much with the findings of Yih and Li [9]. They also agree with the results of Rosenblat and Tanaka [8]) and Bhadauria and Bhatia [17], who found that the effect of the out of phase modulation of the temperature field is stabilizing. Comparing the results of Rosenblat et. al. [8] and Bhadauria and Bhatia [17] one can see that their curves are smooth while here we have an alternative S-curve and H-curve. Their unmodulated case ($\varepsilon = 0$) corresponds to $b = 0$ of this study. Initially, as b increases we have a smooth curve upto $b = 0.6$ and then an alternative H-curve and S-curve. As we go beyond a certain value of b we find a destabilizing effect, however in their cases the stabilizing effect is continuously increasing as ε increases. This difference may be because here the boundary conditions are slightly different from theirs. The above results also agree with the findings of Venezian [5] and Bhatia and Bhadauria [15], who found that for out of phase modulation, initially the effect of the unsteady part is stabilizing but becomes destabilizing at a particular P as ω increases. The results agree with that of Yih and Li [9], who found while studying the instability of an unsteady flow that the effect of modulation is destabilizing. They also found that the critical curve is composed of the two curves, one corresponding to the synchronous solution and the other corresponds to subharmonic one.

Figure 3 shows the variation of the critical wavenumber a corresponding to the critical Rayleigh number R_c in Figure 2. The critical value appears to be discontinuous from the a_c-b curves. This is because in Fig. 2, the S-curves and H-curves are not continued beyond their intersections. The a_c-b curves are also composed of S-curves and H-curves. The existence of synchronous and subharmonic disturbances has already been indicated by Gresho and Sani [10], Yih and Li [9], Clever et al. [22] and Aniss et al. [14] in their investigations.

Figure 4 depicts the value of the Rayleigh number R_{neut} (at neutral stability) with respect to b at 3.1. Therefore the value of R_{neut} is slightly greater than the value of R_c in Figure 2. However the values of R in both the figures do not differ much, as the value of a in Fig. 3 is never very much different from 3.1.

In Fig. 5 we depict the value of R_{neut} with ω at P = 0.73 and b = 0.5. We find that the effect of modulation is stabilizing, decreasing with increasing frequency ω. As ω increases to infinity, the effect of the unsteady part of the primary temperature field disappears altogether. These results are in agreement with Bhatia and Bhadauria [15], Bhadauria and Bhatia [17], Rosenblat and Tanaka [8] and Venezian [5].

Figures 6 and 7 correspond to the variations of R_{neut} with a and P, respectively. From Fig. 6 it is very clear that the critical value of a is very close to 3.1. Figure 7 shows that the enhancement of R_{neut} is greatest when P = 0.6. This means that the effect of modulation is highest stabilizing when P = 0.6.

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